Public Choice manuscript No.
(will be inserted by the editor)

Susceptibilty to coalitional strategic sponsoring

The case of parliamentary agendas

Sébastien Courtin[1], **Boniface Mbih**[1], **Issofa Moyouwou**[2]

[1] CREM UMR CNRS 6211, Faculté de Sciences Economiques et de Gestion, Université de Caen e-mail: boniface.mbih@unicaen.fr
[2] Ecole Normale Supérieure, B.P. 47, Yaoundé, Cameroun

The date of receipt and acceptance will be inserted by the editor

Abstract It usually happens that the alternatives to be voted on in committees are chosen or sponsored by some particularly active committee members. For example, in parliaments, some representatives and some government members are known to be especially active in introducing bills on which the whole committee will later vote. It appears that parliamentary agendas - namely amendment and successive elimination voting rules - are vulnerable to strategic behavior by groups of individuals introducing motions which are not their most preferred alternatives. Our aim in this paper is to evaluate how frequently this type of behavior is susceptible to arise.

Key words Parliamentary agendas - Sponsoring - Strategic behavior - Impartial anonymous culture.

JEL Classification D71.

1 Introduction

Most of the literature on strategic voting studies the manipulation of individual preferences given the issue, that is the set of feasible alternatives. The main result on this topic is the Gibbard-Satterthwaite theorem (Gibbard 1973 and Satterthwaite 1975). It states that every nondictatorial voting procedure selecting a unique outcome is vulnerable to strategic manipulation of preferences, in the sense that one can find some configuration of

Send offprint requests to: Boniface Mbih
Correspondence to: Boniface Mbih, Faculté de Sciences Economiques et de Gestion, Université de Caen, 14032 Caen Cedex.

preferences at which some individual has an incentive to misrepresent her preferences.

Besides, some authors are concerned with a related but different line of investigation, which takes account of a *variable* issue; in other words the set of alternatives open to the vote is not given *a priori*. This line must be distinguished from the literature on the strategic effects of agenda manipulation, which takes into consideration the strategic ordering of alternatives (see Banks 1985; Miller 1995).

One aspect of the variable issue analysis is *strategic candidacy*. More precisely, in an electoral context, strategic candidacy concerns the opportunity for a potential candidate who cannot win the election given the preferences of the voters, to opt in or to opt out, in order to secure another candidate she ranks higher in her preferences than the one who would win the election.

Recently, Osborne and Slivinski (1996) and Besley and Coate (1997) independently develop, in the context of representative democracy, the same basic model of electoral competition in which the political process is modeled as a three-stage game; they further assume that running as a candidate in the election is costly. In stage 1, each citizen decides whether or not to become a candidate; in the second stage, all citizens vote over the set of declared candidates; and in stage 3, the winner can implement her favorite policy and receive benefit from her position. Their main result states that there exist situations in which candidates, with no chance of winning, enter the election in order to affect the identity of the winner, even though such entry is both optional and costly.

In this same context, another important work is due to Dutta, Jackson and Le Breton (2000, 2001); they particularly focus on the incentive of candidates to strategically affect the outcome of a voting procedure. They give specific results for voting procedures based on sequential pairwise elimination of alternatives, but they also provide more general results. They show that the outcome of every nondictatorial voting rule satisfying a unanimity condition is susceptible to be affected by strategic candidacy, not only when the set of voters and candidates are distinct, but also when they overlap.

Another aspect of the variable issue approach, early introduced by Majumdar (1956), and later extensively discussed by Dutta and Pattanaik (1978), is the problem of *strategic sponsoring* of alternatives. In a famous example (see Section 2), Majumdar considers seven voters (the choosers) two of whom are sponsors (two political opponents or two especially energetic members of a committee who usually move all resolutions), and he assumes that all voters always vote according to their sincere preferences. This process of sponsoring takes place in two steps. First, sponsors choose the issue, and second the alternatives in the chosen issue are then submitted to the vote. The example shows that under simple majority rule, it can be advantageous for sponsors to select alternatives non sincerely, i.e. alternatives that do not come highest in their preference orderings.

Dutta and Pattanaik (1978) provide a generalization of the result of Majumdar. They prove that under a large class of voting procedures some voting situations will give individuals opportunities of strategic sponsoring.

One can easily imagine actual situations where the process of sponsoring takes place. The intuition behind the mechanisms we describe mainly refer to parliamentary agendas: in parliaments, some representatives and some government members introduce bills on which the whole committee will later vote. But many other examples can be found in various fields. This is the case for the pre-selection of candidates applying for a position in a firm or some other organization (e.g. lecturers in universities); it usually happens that a limited set of reporters chooses among candidates, and those who pass this step then constitute the set of alternatives submitted to the vote of all committee members. Another example can be found in the area of sports; for the election of the "Ballon d'or" of the journal "France Football", some football - or soccer - players are first selected by a limited set of French journalists and then there is a final vote by a larger international set of journalists. A similar but more sophisticated process can be found for the selection of the "NBA All-Star Games" players.

Also note that sponsoring of alternatives must be distinguished from the rule of k names where “given a set of alternatives a committee chooses k members from this set by voting and make a list with their names, then a single individual from outside the committee selects one of the list names for the office” (see Barbera and Coelho 2004).

In this paper, we examine strategic sponsoring under two families of parliamentary agendas widely studied in the literature (see Rasch 1995; Dutta, Jackson and Le Breton 2001; Mbih, Moyouwou and Picot 2008 among others). We now describe them.

Under these two rules, alternatives are ranked according to a predetermined order, say $a_1a_2a_3$ in the three-alternative case. These rules are commonly used in Parliaments for votes on ordinary motions.

On the one hand, the Anglo-American system is based on a "two by two" procedure, namely the *amendment procedure*: the first ballot is taken between a_1 and a_2 in a pairwise majority contest, and the winner is taken against a_3;

On the other hand, the *successive elimination* rule is used in most countries of Western Europe. Alternatives are considered "one by one": the first vote is on alternatives a_1; if a_1 wins a majority, the procedure terminates; if a_1 is beaten, there is a second vote on a_2, and so on, until there is a winner or a unique alternative left. Note that this amounts to first having a majority contest between a_1 and $\{a_2, a_3\}$, and then between a_2 and a_3 if necessary.

Parliamentary voting procedures are vulnerable to strategic sponsoring of alternatives (see Section 2). However, it remains to evaluate how frequently this behavior is susceptible to occur. As pointed out by Pattanaik (1978, p. 187)

"This is important. For, if the likelihood of such strategic voting is negligible, then one need not be unduly worried about the existence of the possibility as such."

In this contribution, for each of the procedures described above, we are interested in the quantitative significance of the possibilities of strategic sponsoring. Our evaluation relies on an analytical method, as opposed to computer simulations. We compute the exact frequency of strategic sponsoring opportunities under the impartial anonymous culture (IAC), a hypothesis introduced in social choice theory by Kuga and Nagatani (1974) and later developed by Gehrlein and Fishburn (1976). IAC is based on the assumption that all anonymous profiles (see Section 2) have the same probability of occurrence. In other words, our purpose is to compute the following ratio:

$$\frac{\text{number of anonymous profiles vulnerable to coalitional strategic sponsoring}}{\text{total number of all possible anonymous profiles}}$$

The remainder of the paper is organized as follows: Section 2 is a presentation of the general framework, with definitions, assumptions and some examples; Section 3 gives a characterization of coalitional strategic sponsoring situations; next, Section 4 provides our main results about strategic sponsoring; and finally Section 5 concludes the paper.

2 Notations and definitions

Consider a finite set N of n voters (choosers), with $n \geq 3$, and a finite set $\mathbb{S}$ of σ sponsors, with $\sigma \geq 3$. A will denote the set of alternatives, and 2^A the set of all possible nonempty subsets of A. Further, B will denote the issue, that is the set of sponsored alternatives, $B \in 2^A$.

The process of sponsoring, which leads to final outcomes, takes place in two distinct stages. First, sponsors choose, within the set A, the alternatives that will be submitted to the vote at stage two. Each sponsor *independently* chooses one alternative within this set, and one same alternative can be chosen by more than one sponsor. The issue B is determined as the outcome of the choice of alternatives by sponsors. At stage two the voters express their individual preference orderings over B. And given a voting procedure and the preference orderings expressed by individuals over the issue, the final outcome is determined.

Notice that in real life not all individuals may be potential sponsors (for example, in parliaments, bills are only introduced by some active representatives or government members). We shall thus admit that $\sigma \leq n$. Moreover we assume that the sets of voters and sponsors are disjoint, i.e. $\mathbb{S} \cap N = \varnothing$. It was also possible to consider the case where $\mathbb{S} \subset N$. This distinction can be illustrated by the usual ways laws are introduced in Parliaments. For example in France (and many other countries), bills are proposed by ministers - who do not participate to the vote - while private bills are proposed by representatives - who participate to the vote. In our study, in order to avoid unnecessary complexities, we only consider the case $\mathbb{S} \cap N = \varnothing$, since with a fixed number of sponsors and a large electorate, both cases lead to very similar results.

Let L be the set of all possible linear orderings on A, that is the set of all complete, antisymmetric and transitive binary relations on A. R^i denotes voter i's preference relation and $R^i \in L$. A profile of voters' preferences is an n-tuple $R^N = (R^1, R^2, ..., R^n)$ of individual preference relations, one for each individual voter, and L^N is the set of all such profiles. In the same way for each sponsor j, let R^j be $j's$ preference relation; then $R^{\mathbb{S}}$ is a profile of sponsors' preferences, and $L^{\mathbb{S}}$ is the set of all such profiles.

We now present the strategic sponsoring behavior more formally. We first need some general definitions.

Definition 1 *A social choice function (SCF) is a mapping f from L^N to A.*

Including a sponsoring process leads to a notion which is more general than an *SCF*. For each profile $R^{\mathbb{S}}$ of sponsors preferences over the set A of alternatives, one issue $B \in 2^A$ is selected by the choice of a single alternative by each sponsor. And for this issue, preferences of the voters determine the final outcome.

Definition 2 *A social choice function with sponsoring (SCFS) is a mapping g from $2^A \times L^{\mathbb{S}} \times L^N$ to A, such that for all $\left(B, R^{\mathbb{S}}, R^N\right) \in 2^A \times L^{\mathbb{S}} \times L^N$*

$$g\left(B, R^{\mathbb{S}}, R^N\right) \in B.$$

Let $\bar{B}$ be the set of sponsored alternatives when sponsors choose sincerely, that is when each of them chooses her most preferred alternative. $\bar{B}$ is called a sincere issue.

Definition 3 *Let S be the set of sponsors and $\bar{B}$ the sincere issue. Given an SCFS g, a profile $\left(R^{\mathbb{S}}, R^N\right)$ is unstable via coalitional strategic sponsoring if there exist some nonempty subset S' of S and some other issue B', such that*

(i) $\bar{B} \cap B' \neq \varnothing$;
(ii) $g\left(\bar{B}, R^{\mathbb{S}}, R^N\right) \neq g\left(B', R^{\mathbb{S}}, R^N\right)$;
(iii) and $g\left(B', R^{\mathbb{S}}, R^N\right) R^j g\left(\bar{B}, R^{\mathbb{S}}, R^N\right)$ *for all* $j \in \mathbb{S}'$.

In words, strategic sponsoring occurs if there exists some group of sponsors S' (possibly a single sponsor), who have an incentive to sponsor another alternative in order to ensure that the new outcome is better from their viewpoint.

Definition 4 *A social choice function with sponsoring g is vulnerable to coalitional strategic sponsoring if there exists at least one profile $\left(\bar{B}, R^{\mathbb{S}}, R^N\right)$ unstable via coalitional strategic sponsoring.*

In the sequel, where there is no ambiguity we shall simply write unstable *via* strategic sponsoring.

Note that a society choosing an alternative from a finite set using an SCFS can be viewed as a game in normal form where (i) the set of players is the whole set of agents, sponsors and voters, although we suppose here that only sponsors can behave strategically; (ii) the set of strategies open to agents is the set of all linear orders on A, and (iii) the payoff function is the SCFS. With this interpretation of our context, an unstable situation via coalitional strategic sponsoring appears as a profile that *is not a strong equilibrium point* of the given game.

To illustrate the strategic sponsoring process, let us now present the (famous) example of Majumdar (1956).

Example 1 Suppose there are two sponsors 1 and 2 among seven voters; further there are four alternatives, a_1, a_2, a_3 and a_4, and the *SCFS* is the plurality rule. Note that, in contrast to our study Majumdar assumes that the sponsors participate to the vote. Individual preferences are as follows:

1	2	3	4	5	6	7
a_4	a_3	a_1	a_3	a_3	a_2	a_4
a_1	a_2	a_2	a_4	a_1	a_1	a_2
a_2	a_1	a_3	a_2	a_4	a_3	a_1
a_3	a_4	a_4	a_1	a_2	a_4	a_3

If we suppose that the sponsors choose sincerely, then the sincere issue is $\bar{B} = \{a_3, a_4\}$. Then the voters express their individual preference orderings over $\bar{B}$, and a_3 is chosen. But if sponsor 1 decides to sponsor a_1 instead of a_4, then the new issue is $B^{'} = \{a_1, a_3\}$, and a_1 is chosen. And since sponsor 1 prefers a_1 to a_3, it appears that the profile above is unstable *via* strategic sponsoring.

Up to now, all definitions and the example above have been given in terms of profiles, but only for convenience. In the remainder of this work, we will be interested only in anonymous profiles, in the sense that we do not distinguish between two profiles that differ only by the identity of the individuals who express the same preference relation. In other words, the outcome of a given configuration of individual preferences only depends on the number of individuals with some type of preference relation, and not on which individuals have that type of preferences. Thus, we assume anonymity over the set of voters and over the set of sponsors.

From now on, we shall focus on the three-alternative case. Given a set $A = \{a_1, a_2, a_3\}$, there are exactly six linear orderings on A, labeled below:

$R_1 : a_1a_2a_3$, $R_2 : a_1a_3a_2$, $R_3 : a_2a_1a_3$, $R_4 : a_2a_3a_1$, $R_5 : a_3a_1a_2$, $R_6 : a_3a_2a_1$.

Then, a *voting situation* $\mathfrak{n}$ is an anonymous voting profile obtained from a profile R^N by rewriting it as $\mathfrak{n} = (n_1, n_2, n_3, n_4, n_5, n_6)$, where for each

$k = 1, \ldots, 6$, n_k is the number of voters in N with preference relation R_k. A situation is thus a 6−tuple of natural integers such that $\Sigma_{k=1}^{6} n_k = n$. The set of all voting situations will be denoted $\mathfrak{N}$. Given a voting situation $\mathfrak{n} \in \mathfrak{N}$ and two alternatives a_j and a_k, $n(a_j, a_k, \mathfrak{n})$ is the number of voters who prefer a_j to a_k given situation $\mathfrak{n}$.

Likewise, a *sponsoring situation* $s = (s_1, s_2, s_3, s_4, s_5, s_6)$ is obtained from a profile $R^{\mathbb{S}}$ where s_k is the number of individuals in S with preference relation R_k and $\Sigma_{k=1}^{6} s_k = \sigma$. The set of all sponsoring situations will be denoted S. And $\sigma(a_j, a_k, s)$ is the number of sponsors who prefer a_j to a_k given situation s.

Note that every profile $\left(R^{\mathbb{S}}, R^N\right)$ can be rewritten as a situation $(s, \mathfrak{n})$; $(s, \mathfrak{n})$ is then said to be associated with $\left(R^{\mathbb{S}}, R^N\right)$. The definition below follows straightforwardly from Definitions 3 and 4.

Definition 5 *A situation is unstable (via coalitional strategic sponsoring) if it is associated with an unstable profile.*

Definition 6 *A social choice function with sponsoring g, is vulnerable to coalitional strategic sponsoring if there exists some unstable situation under g.*

In our study we consider two families of parliamentary agendas, namely the amendment and successive elimination procedures, with possibly qualified majority. An α-majority contest, first introduced in the social choice literature by Slutsky (1979), is a rule under which, given some $\alpha \in \,]0, 1[$ and a voting situation $\mathfrak{n}$, an alternative a_h is socially preferred to (at least as good as) a_k if the number of individuals who prefer a_h to a_k is at least $\frac{\alpha}{1-\alpha}$ times the number of individuals who prefer a_k to a_h. Moreover, ties are broken in favor of the alternatives with the greatest index, which can be written as follows:

$$a_h \text{ beats } a_k \Leftrightarrow \begin{cases} h < k \Rightarrow n(a_k, a_h, \mathfrak{n}) < \alpha n(\mathfrak{n}) \\ k < h \Rightarrow n(a_h, a_k, \mathfrak{n}) \geq \alpha n(\mathfrak{n}) \end{cases}$$

Notice that simple majority corresponds to $\alpha = \frac{1}{2}$. Also note that small values of α give a significant advantage to alternatives with greater indices. Reciprocally, large values of α give an advantage to alternatives with smaller indices.

Under the α-amendment procedure (denoted AP_α) one assigns to a voting situation $\mathfrak{n}$ the alternative $AP_\alpha(B, s, \mathfrak{n})$ defined as follows:

(a) if B contains a single alternative x, then $AP_\alpha(B, s, \mathfrak{n}) = x$;
(b) if B is a pair of two alternatives x and y, then $AP_\alpha(B, s, \mathfrak{n})$ is the winner of the α-majority contest between x and y;
(c) if $B = \{a_1, a_2, a_3\}$ then $AP_\alpha(B, s, \mathfrak{n})$ is selected from the α-majority contest between a_3 and the winner of the α-majority contest between a_1 and a_2. Then, only one of the four scenarios below holds:

a_1 beats a_2 at the first ballot and a_1 beats a_3 at the second ballot

$$a_1 \text{ beats } a_2 \text{ at the first ballot and } a_3 \text{ beats } a_1 \text{ at the second ballot}$$

$$a_2 \text{ beats } a_1 \text{ at the first ballot and } a_2 \text{ beats } a_3 \text{ at the second ballot}$$

$$a_2 \text{ beats } a_1 \text{ at the first ballot and } a_3 \text{ beats } a_2 \text{ at the second ballot}$$

Under the α-successive elimination procedure (denoted SE_α), one assigns to a voting situation $\mathfrak{n}$ the alternative $SE_\alpha(B, s, \mathfrak{n})$ defined as follows:

(i) if B contains at most two alternatives then $SE_\alpha(B, s, \mathfrak{n})$ and $AP_\alpha(B, s, \mathfrak{n})$ coincide;
(ii) for $B = \{a_1, a_2, a_3\}$, voters have to compare $A_1 = \{a_1\}$ and $A_2 = \{a_2,\ a_3\}$ at the first ballot. If A_1 is selected then $SE_\alpha(B, s, \mathfrak{n}) = a_1$, otherwise $SE_\alpha(B, s, \mathfrak{n})$ is the winner of the α-majority contest between a_2 and a_3 at the second ballot. Then, only one of the three scenarios below holds:

$$a_1 \text{ beats } \{a_2,\ a_3\} \text{ at the first ballot}$$

$$\{a_2,\ a_3\} \text{ beats } a_1 \text{ at the first ballot and } a_2 \text{ beats } a_3 \text{ at the second ballot}$$

$$\{a_2,\ a_3\} \text{ beats } a_1 \text{ at the first ballot and } a_3 \text{ beats } a_2 \text{ at the second ballot}$$

Further, for the latter class of procedures, every individual must compare $A_1 = \{a_1\}$ with the set $A_2 = \{a_2,\ a_3\}$. Thus additional information on individual behavior has to be provided, explaining from R^i which subset, out of $A_1 = \{a_1\}$ and $A_2 = \{a_2,\ a_3\}$, is most preferred by voter i. We then assume that there are two types of voters in society N, with two distinct attitudes, namely maximin behavior or maximax behavior.

Then, with the successive elimination rules, we shall consider two different families of procedures:

(1) the voters are all supposed to have maximin behavior, then we denote this type of successive elimination rules, SEm_α;

(2) and the voters are all supposed to have maximax behavior, then we denote this type of successive elimination rules, SEM_α.

The reader can easily check that these two families of rules are vulnerable to strategic sponsoring, as illustrated by the following example.

Example 2 Consider three sponsors who do not participate to the vote, five voters, and alternatives a_1, a_2 and a_3. $AP_{\frac{1}{2}}$ is the voting rule. Assume individual preferences are as follows:

	Sponsors		
R_k	R_1	R_3	R_4
	a_1	a_2	a_2
	a_3	a_1	a_3
	a_2	a_3	a_1
s_k	1	1	1

	Voters			
R_k	R_1	R_3	R_5	R_6
	a_1	a_2	a_3	a_3
	a_2	a_1	a_1	a_2
	a_3	a_3	a_2	a_1
s_k	1	1	2	1

A sincere behavior by the sponsors leads to $\bar{B} = \{a_1, a_2\}$ and $AP_{\frac{1}{2}}(\bar{B}, s, \mathfrak{n}) = a_1$. However if sponsor with preference R_3 chooses a_3 rather than a_2, then the new issue is $B' = \{a_1, a_2, a_3\}$ and $AP_{\frac{1}{2}}(B', s, \mathfrak{n}) = a_3$. We observe that a_3 is preferred to a_1 by that sponsor.

Our main purpose in this contribution to social choice theory is to compare the three families of rules described above through their vulnerability to strategic sponsoring, for any possible situation. In order to do that, we compute the ratio

$$\frac{\text{number of situations } (s, \mathfrak{n}) \text{ unstable } \textit{via} \text{ coalitional strategic sponsoring}}{\text{total number of all possible situations } (s, \mathfrak{n})}$$

Note that such frequency evaluations are based on the hypothesis that all possible situations $(s, \mathfrak{n})$ have the same probability of occurrence. Thus, as said above, we assume anonymity over the set of voters and over the set of sponsors.

In the next two sections, we characterize situations at which for each voting system considered in this paper coalitional strategic sponsoring is susceptible to occur. And these characterizations are then used to obtain formulae giving frequencies measuring the vulnerability of those systems to strategic sponsoring.

3 Characterizations of unstable situations

As explained in Definition 6, a social choice function with sponsoring g, is vulnerable to coalitional strategic sponsoring if there exists some unstable situation under g. This section characterizes the sets of all situations $(s, \mathfrak{n})$ unstable *via* strategic sponsoring. More precisely, to illustrate the reasoning, we focus on the particular case where a_1 is the initial winner. For a general characterization of vulnerable situation for amendment and successive elimination procedures with maximin and maximax see the associated working paper (Courtin, Mbih and Moyouwou 2008). Likewise, not all proofs are given below, because they are very similar; we only provide details for the proof concerning the amendment procedure. For all the other proofs, the reader can refer to the associated working paper.

3.1 Preliminary observations on strategic sponsoring

Given a set of three distinct alternatives $\{a_h, a_j, a_k\}$, let us consider a social choice function with sponsors. We assume that sponsors can choose only one alternative. Depending on sponsors preferences, several sincere issues $\bar{B}$ are conceivable. Moreover, according to a particular sincere issue, strategic behavior by the sponsors can occur in several ways.

First suppose that $\bar{B}$ consists of a unique alternative a_h, that is all sponsors have a_h as the same most preferred alternative. Then by the definition of a social choice function with sponsors, a_h is selected by the voters. Since a_h is trivially the best outcome for each sponsor, therefore there is clearly no way for strategic sponsoring.

Now suppose that the sincere issue is $\bar{B} = \{a_h, a_j\}$ and without loss of generality, assume that a_h is elected. Then there are three types of possible collective strategic sponsoring: the new issue is $\{a_h, a_k\}$ and a_k is finally elected (Type 1); the new issue is $\{a_h, a_j, a_k\}$ and a_j wins (Type 2); and the new issue is $\{a_h, a_j, a_k\}$ and a_k wins (Type 3). Note that strategic behavior in case Type 3 is undertaken in favor of alternative a_k, which is added to the sincere issue $\bar{B}$, while in case Type 2, strategic sponsoring is in favor of a_j, which was already sponsored in $\bar{B}$. Also observe that, for strategic sponsoring to occur, the case where the new issue is $\{a_h\}$ is not possible since a_h is already the winner. In the same way, from $\bar{B} = \{a_h, a_j\}$, situations where the new issue is $\{a_j\}$ or $\{a_k\}$ or $\{a_j, a_k\}$ are not rational, since this would require a change by a_h's sponsors, which is not possible since a_h is their most preferred alternative.

Finally, suppose that $\bar{B} = \{a_h, a_j, a_k\}$ and that a_h is elected. Then there is a unique way for strategic sponsoring to occur, Type 4, that is the new issue is $\{a_h, a_j\}$ and a_j is elected (or $\{a_h, a_k\}$ is the new issue and a_k wins). For the same reason as above, situations where the new issue is $\{a_h\}$, $\{a_j\}$, $\{a_k\}$ or $\{a_j, a_k\}$ are not feasible.

In the next subsection and for each social choice function with sponsoring under study, strategic attitudes Type 1, Type 2, Type 3 and Type 4 will be useful in the proofs of characterizations results.

3.2 Unstable situations via strategic sponsoring

The statement below describes how a situation $(s, \mathfrak{n})$ must be in order for it to be subject to possible strategic sponsoring. For simplicity, we only consider the case where a_1 is elected.

Proposition 1 *Suppose AP_α is the voting rule and a_1 is elected given some situation $(s, \mathfrak{n})$. Then $(s, \mathfrak{n})$ is unstable via coalitional strategic sponsoring if and only if*

(1.a)
$$\begin{cases} n_1+n_2+n_5>n-\alpha n & (1.1)\\ n_4+n_5+n_6\geq\alpha n & (1.2)\\ s_1+s_2\geq 1 & (1.3)\\ s_4\geq 1 & (1.4)\\ s_1+s_2+s_4=\sigma & (1.5)\end{cases}$$

or (1.d)
$$\begin{cases} n_1+n_2+n_3>n-\alpha n & (1.18)\\ n_3+n_4+n_6\geq\alpha n & (1.19)\\ s_1+s_2\geq 1 & (1.20)\\ s_6\geq 1 & (1.21)\\ s_1+s_2+s_6=\sigma & (1.22)\end{cases}$$

or (1.b)

$$\begin{cases} n_1+n_2+n_5>n-\alpha n & (1.6)\\ n_4+n_5+n_6\geq\alpha n & (1.7)\\ s_1+s_2\geq 1 & (1.8)\\ s_3+s_4\geq 2 & (1.9)\\ s_4\geq 1 & (1.10)\\ s_1+s_2+s_3+s_4=\sigma & (1.11)\end{cases}$$

or (1.e)

$$\begin{cases} n_1+n_2+n_3>n-\alpha n & (1.23)\\ n_3+n_4+n_6\geq\alpha n & (1.24)\\ n_1+n_3+n_4>n-\alpha n & (1.25)\\ s_1+s_2\geq 1 & (1.26)\\ s_5+s_6\geq 2 & (1.27)\\ s_6\geq 1 & (1.28)\\ s_1+s_2+s_5+s_6=\sigma & (1.29)\end{cases}$$

or (1.c)

$$\begin{cases} n_1+n_2+n_3>n-\alpha n & (1.12)\\ n_3+n_4+n_6\geq\alpha n & (1.13)\\ n_2+n_5+n_6\geq\alpha n & (1.14)\\ s_1+s_2\geq 1 & (1.15)\\ s_5+s_6\geq 2 & (1.16)\\ s_1+s_2+s_5+s_6=\sigma & (1.17)\end{cases}$$

Proof Necessity. Let $(s, \mathfrak{n})$ be an unstable situation and suppose that a_1 is elected. From the previous analysis, the sincere issue is $\{a_1, a_2\}$, $\{a_1, a_3\}$ or $\{a_1, a_2, a_3\}$.

First suppose that the sincere issue $\bar{B} = \{a_1, a_2\}$. Only strategic sponsoring Type 1, Type 2 or Type 3 are conceivable.

Type 1: the new issue is $\{a_1, a_3\}$ and strategic sponsoring initiated by the sponsors of a_2 is in favor of a_3 (1.a). Therefore there is at least one sponsor with preference $a_1a_2a_3$ or $a_1a_3a_2$ (1.3), at least one sponsor with preference $a_2a_3a_1$ (1.4) and no sponsor with preference $a_2a_1a_3$ ($s_3 = 0$ in (1.5) (since such a sponsor has no incentive to favor a_3). Also observe that since $\bar{B} = \{a_1, a_2\}$, no sponsor has preference $a_3a_1a_2$ nor $a_3a_2a_1$ ($s_5 = s_6 = 0$ in 1.5). Given the sincere issue $\bar{B}$, a_1 beats a_2 (1.1) and given the new issue $\{a_1, a_3\}$, a_3 beats a_1 (1.2).

Type 2: the new issue is $\{a_1, a_2, a_3\}$ and strategic sponsoring is initiated by the sponsors of a_2 in favor of a_3 (1.b). Then there is at least one sponsor with preference $a_1a_2a_3$ or $a_1a_3a_2$ (1.8), at least two sponsors for a_2 (1.9) with at least one sponsor with preference $a_2a_3a_1$ (1.10). And since $\bar{B} = \{a_1, a_2\}$, no sponsor has preference $a_3a_1a_2$ or $a_3a_2a_1$ ($s_5 = s_6 = 0$ in 1.11). According to the sincere issue $\bar{B}$, a_1 beats a_2 (1.6) and given the new issue $\{a_1, a_2, a_3\}$, a_1 beats a_2 (1.6) and a_3 beats a_1 (1.7), (note that at the first ballot a_1 beats necessarily a_2).

Type 3: the new issue is $\{a_1, a_2, a_3\}$ and strategic sponsoring must be in favor of a_2. However a_1 beats again a_2 at the first ballot. Therefore strategic sponsoring Type 3 does not occur under the amendment procedure. Note that this is a consequence of the predetermined order $a_1a_2a_3$ of pairwise majority contests.

Now assume that the sincere issue $\bar{B} = \{a_1, a_3\}$. Only strategic sponsoring Type 1, Type 2 or Type 3 are conceivable. For Type 1 (1.d) and type 2

(1.e), the proof is similar to the one above (just replace a_2 with a_3 and a_3 with a_2).

Type 3: the new issue is $\{a_1, a_2, a_3\}$ and strategic sponsoring initiated by the sponsors of a_3 is in favor of a_3 (1.c). Then there is at least one sponsor with preference $a_1a_2a_3$ or $a_1a_3a_2$ (1.15), at least two sponsors for a_3 (1.16) and no sponsor with preference $a_2a_1a_3$ nor $a_2a_3a_1$ ($s_3 = s_4 = 0$ in (1.17). Given the sincere issue $\bar{B}$, a_1 beats a_3 (1.12) and given the new issue $\{a_1, a_2, a_3\}$, a_2 must beat a_1 at the first ballot (1.13); if not, a_1 will defeat a_3 at the second ballot. And a_3 beats a_2 at the second ballot (1.14).

Finally, assume $\bar{B} = \{a_1, a_2, a_3\}$. Then only strategic sponsoring Type 4 is conceivable.

Type 4: the new issues must be $\{a_1, a_2\}$ or $\{a_1, a_3\}$, but a_1 beats both a_2 and a_3. Therefore there is no strategic sponsoring Type 4 under the amendment procedure.

Sufficiency. It is straightforward that strategic sponsoring Type 1 occurs under constraints (1.a) or (1.d); strategic sponsoring Type 2 occurs under constraints (1.b) or (1.e); and strategic sponsoring Type 3 occurs under constraints (1.c).

We then consider successive elimination with maximin behavior.

Proposition 2 *Suppose SEm_α is the voting rule and a_1 is elected given some situation $(s, \mathfrak{n})$. Then $(s, \mathfrak{n})$ is unstable via coalitional strategic sponsoring if and only if*

(2.a)
$$\begin{cases} n_1+n_2+n_5>n-\alpha n \\ n_4+n_5+n_6\geq\alpha n \\ s_1+s_2\geq1 \\ s_4\geq1 \\ s_1+s_2+s_4=\sigma \end{cases}$$

or (2.b)
$$\begin{cases} n_1+n_2+n_3>n-\alpha n \\ n_3+n_4+n_6\geq\alpha n \\ s_1+s_2\geq1 \\ s_6\geq1 \\ s_1+s_2+s_6=\sigma \end{cases}$$

or (2.c)
$$\begin{cases} n_1+n_2+n_3+n_5>n-\alpha n \\ n_3+n_4+n_6\geq\alpha n \\ s_1+s_2\geq1 \\ s_3+s_4\geq1 \\ s_6\geq1 \\ s_1+s_2+s_3+s_4+s_6=\sigma \end{cases}$$

or (2.d)
$$\begin{cases} n_1+n_2+n_3+n_5>n-\alpha n \\ n_4+n_5+n_6\geq\alpha n \\ s_1+s_2\geq1 \\ s_5+s_6\geq1 \\ s_4\geq1 \\ s_1+s_2+s_4+s_5+s_6=\sigma \end{cases}$$

And finally, successive elimination with maximax behavior.

Proposition 3 *Suppose SEM_α is the voting rule and a_1 is elected given some situation $(s, \mathfrak{n})$. Then $(s, \mathfrak{n})$ is unstable via coalitional strategic sponsoring if and only if*

(3.a)
$$\begin{cases} n_1+n_2+n_5>n-\alpha n \\ n_4+n_5+n_6\geq\alpha n \\ s_1+s_2\geq 1 \\ s_4\geq 1 \\ s_1+s_2+s_4=\sigma \end{cases}$$

or (3.b)
$$\begin{cases} n_1+n_2+n_3>n-\alpha n \\ n_3+n_4+n_6\geq\alpha n \\ s_1+s_2\geq 1 \\ s_6\geq 1 \\ s_1+s_2+s_6=\sigma \end{cases}$$

or (3.c)
$$\begin{cases} n_1+n_2+n_5>n-\alpha n \\ n_3+n_4+n_5+n_6\geq\alpha n \\ n_2+n_5+n_6\geq\alpha n \\ s_1+s_2\geq 1 \\ s_3+s_4\geq 2 \\ s_4\geq 1 \\ s_1+s_2+s_3+s_4=\sigma \end{cases}$$

or (3.d)
$$\begin{cases} n_1+n_2+n_3>n-\alpha n \\ n_3+n_4+n_5+n_6\geq\alpha n \\ n_1+n_3+n_4>n-\alpha n \\ s_1+s_2\geq 1 \\ s_5+s_6\geq 2 \\ s_6\geq 1 \\ s_1+s_2+s_5+s_6=\sigma \end{cases}$$

or (3.e)
$$\begin{cases} n_1+n_2+n_5>n-\alpha n \\ n_3+n_4+n_5+n_6\geq\alpha n \\ n_1+n_3+n_4>n-\alpha n \\ s_1+s_2\geq 1 \\ s_3+s_4\geq 2 \\ s_1+s_2+s_3+s_4=\sigma \end{cases}$$

or(3.f)
$$\begin{cases} n_1+n_2+n_3>n-\alpha n \\ n_3+n_4+n_5+n_6\geq\alpha n \\ n_2+n_5+n_6\geq\alpha n \\ s_1+s_2\geq 1 \\ s_5+s_6\geq 2 \\ s_1+s_2+s_5+s_6=\sigma \end{cases}$$

In the next section, these characterizations will be subsequently used to compute the likelihood of strategic sponsoring opportunities.

4 Strategic sponsoring occurrence

In this section and for each of the rules under study, two series of results are provided: (*i*) we first give formulae stating, for $\alpha = \frac{1}{2}$, the frequencies of coalitional strategic sponsoring with respect to n and σ and (*ii*) we then give general formulae stating those frequencies as a function of α, n and σ. We also consider special cases, specifically when the electorate is infinitely large or when the number of sponsors is arbitrarily fixed. All these results are derived from the characterizations of unstable situations by the use of computerised evaluations processes based on the same technique as the one in Gehrlein and Fishburn (1976), Gehrlein and Lepelley (1999) or Huang and Chua (2000).

Let us recall that the total number of situations $(s, \mathfrak{n})$ with three alternatives is $\binom{\sigma+5}{5}\binom{n+5}{5} = \frac{(\sigma+1)(\sigma+2)(\sigma+3)(\sigma+4)(\sigma+5)(n+1)(n+2)(n+3)(n+4)(n+5)}{14\,400}$

4.1 Amendment rules

Let $H(\alpha, n, \sigma)$ be the likelihood that under amendment rules with α-majority contests, a strategic sponsoring situation exists with n voters and σ sponsors. We shall first consider the case $\alpha = \frac{1}{2}$. We obtain the following results:

Proposition 4 *Let $AP_{\frac{1}{2}}$ be the voting rule. Suppose σ is a fixed proportion k of n, that is $\sigma = \frac{n}{k}$. Then for $n \geq 2$, $H(\frac{1}{2}, n, \frac{n}{k}) =$*

$$\begin{cases} \dfrac{5kn\left(\substack{-24k^4n-144k^4-290k^3n^2-1740k^3n-1280k^3+215k^2n^3+\\ 1290k^2n^2+960k^2n+74kn^4+444kn^3+320kn^2+n^5+6n^4}\right)}{16(2k+n)(3k+n)(4k+n)(5k+n)(n+1)(n+5)(k+n)} & \text{if } n \equiv 0 \mod 2 \\[2ex] \dfrac{5k\left(\substack{-24k^4n^2-144k^4n+168k^4-290k^3n^3-1740k^3n^2-1810k^3n+215k^2n^4+\\ 1290k^2n^3+1375k^2n^2+74kn^5+444kn^4+442kn^3+n^6+6n^5-7n^4}\right)}{16(2k+n)(3k+n)(4k+n)(5k+n)(n+2)(n+4)(k+n)} & \text{if } n \equiv 1 \mod 2 \end{cases}$$

For some special values of k, we have:

Corollary 1 *Suppose $AP_{\frac{1}{2}}$ is the voting rule. Then for $n \geq 2$*

$$H(\tfrac{1}{2}, n, \tfrac{n}{5}) = \begin{cases} \dfrac{25n\left(n^5+376n^4+7595n^3-2400n^2-208\,500n-250\,000\right)}{16(n+1)(n+5)^2(n+10)(n+15)(n+20)(n+25)} & \text{if } n \equiv 0 \mod 2 \\[2ex] \dfrac{25\left(n^6+376n^5+7588n^4-1790n^3-198\,125n^2-316\,250n+105\,000\right)}{16(n+2)(n+4)(n+5)(n+10)(n+15)(n+20)(n+25)} & \text{if } n \equiv 1 \mod 2 \end{cases}$$

$$H(\tfrac{1}{2}, n, \tfrac{n}{4}) = \begin{cases} \dfrac{5n\left(n^5+302n^4+5216n^3+3360n^2-102\,144n-118\,784\right)}{4(n+1)(n+4)(n+5)(n+8)(n+12)(n+16)(n+20)} & \text{if } n \equiv 0 \mod 2 \\[2ex] \dfrac{5\left(n^6+302n^5+5209n^4+3848n^3-95\,504n^2-152\,704n+43\,008\right)}{4(n+2)(n+4)^2(n+8)(n+12)(n+16)(n+20)} & \text{if } n \equiv 1 \mod 2 \end{cases}$$

$$H(\tfrac{1}{2}, n, \tfrac{n}{3}) = \begin{cases} \dfrac{15n\left(n^5+228n^4+3267n^3+4740n^2-40\,284n-46\,224\right)}{16(n+1)(n+3)(n+5)(n+6)(n+9)(n+12)(n+15)} & \text{if } n \equiv 0 \mod 2 \\[2ex] \dfrac{15\left(n^6+228n^5+3260n^4+5106n^3-36\,549n^2-60\,534n+13\,608\right)}{16(n+2)(n+3)(n+4)(n+6)(n+9)(n+12)(n+15)} & \text{if } n \equiv 1 \mod 2 \end{cases}$$

$$H(\tfrac{1}{2}, n, \tfrac{n}{2}) = \begin{cases} \dfrac{5n\left(n^5+154n^4+1748n^3+3480n^2-10\,464n-12\,544\right)}{8(n+1)(n+2)(n+4)(n+5)(n+6)(n+8)(n+10)} & \text{if } n \equiv 0 \mod 2 \\[2ex] \dfrac{5\left(n^6+154n^5+1741n^4+3724n^3-8804n^2-16\,784n+2688\right)}{8(n+2)^2(n+4)^2(n+6)(n+8)(n+10)} & \text{if } n \equiv 1 \mod 2 \end{cases}$$

$$H(\tfrac{1}{2}, n, n) = \begin{cases} \dfrac{5n\left(n^5+80n^4+659n^3+1320n^2-804n-1424\right)}{16(n+1)^2(n+2)(n+3)(n+4)(n+5)^2} & \text{if } n \equiv 0 \mod 2 \\[2ex] \dfrac{5(n-1)\left(n^5+81n^4+733n^3+2175n^2+1786n-168\right)}{16(n+1)(n+2)^2(n+3)(n+4)^2(n+5)} & \text{if } n \equiv 1 \mod 2 \end{cases}$$

These results can also be stated graphically in Figure 1:

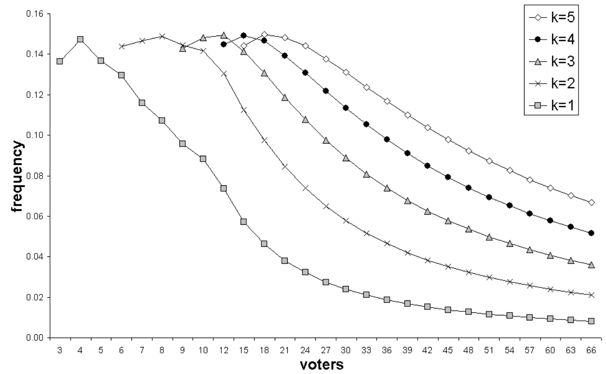

Fig.1 Strategic sponsoring under amendment with $\frac{n}{k}$ sponsors

Note that with any fixed proportion $\frac{n}{k}$ of sponsors, frequencies tend to 0 as the number of voters rises. Also notice that given a fixed number of voters, the smaller the number of sponsors as compared with the number of voters, the higher the frequencies of coalitional strategic sponsoring. For example for $n = 577$, the size of the french parliament, the frequencies are 0.004 for $k = 5$ and 0.001 for $k = 2$. That is for $n = 577$, when the number of sponsors is five times less than the number of voters, strategic sponsoring occurs almost three times more than when the number of sponsors is half the number of voters. Table 1 in Appendix summarizes the results above.

Let us now present the more general formulae corresponding to $\alpha = \frac{1}{2}$.

Proposition 5 *Suppose $AP_{\frac{1}{2}}$ is the voting rule. Then for $n \geq 2$ and $\sigma \geq 3$,*
$H(\frac{1}{2}, n, \sigma) =$

$$\begin{cases} \dfrac{5\begin{pmatrix} n^2\sigma^4+74n^2\sigma^3+215n^2\sigma^2-290n^2\sigma-24n^2+6n\sigma^4+ \\ 444n\sigma^3+1290n\sigma^2-1740n\sigma-144n+320\sigma^3+960\sigma^2-1280\sigma \end{pmatrix}}{16(\sigma+1)(\sigma+2)(\sigma+3)(\sigma+4)(\sigma+5)(n+1)(n+5)} & \textit{if } n \equiv 0 \mod 2 \\ \dfrac{5\begin{pmatrix} n^2\sigma^4+74n^2\sigma^3+215n^2\sigma^2-290n^2\sigma-24n^2+6n\sigma^4+444n\sigma^3+ \\ 1290n\sigma^2-1740n\sigma-144n-7\sigma^4+442\sigma^3+1375\sigma^2-1810\sigma+168 \end{pmatrix}}{16(\sigma+1)(\sigma+2)(\sigma+3)(\sigma+4)(\sigma+5)(n+2)(n+4)} & \textit{if } n \equiv 1 \mod 2 \end{cases}$$

Corollary 2 *In a large society,*

$$H(\tfrac{1}{2}, \infty, \sigma) = \frac{5\left(\sigma^4+74\sigma^3+215\sigma^2-290\sigma-24\right)}{16(\sigma+1)(\sigma+2)(\sigma+3)(\sigma+4)(\sigma+5)}$$

For some special values of σ, we have:

σ (sponsors)	3	4	5	6
$H(\frac{1}{2}, \infty, \sigma)$	0.145	0.150	0.142	0.131

These different limits show that the occurrence of strategic sponsoring, with large value of n, is maximal for 4 sponsors and then decreases as σ rises. Also note that $H(\frac{1}{2}, n, \sigma)$ tends to 0 when σ tends to infinity.

From the characterizations of unstable situations above, we also derive general formulae stating those frequencies as a function of α, n and σ. Since these formulae are not easily interpretable, the reader can refer to the associated working paper. However, Figure 3 below gives an illustration of this type of relations when the number of sponsors and the number of voters are large. We then give simpler formulae below, specifically when the electorate is infinitely large and the number of sponsors is arbitrarily fixed.

Proposition 6 *Suppose that AP_α is the voting rule, σ changes from 3 to 6 and n tends to infinity, then*

$$H(\alpha,\infty,3)=\begin{cases}\frac{180\alpha^3-630\alpha^4+576\alpha^5+7}{56} & \text{if } 0\leq\alpha\leq\frac{1}{2}\\ \frac{-900\alpha+2520\alpha^2-3420\alpha^3+2250\alpha^4-576\alpha^5+133}{56} & \text{if } \frac{1}{2}\leq\alpha<1\end{cases}$$

$$H(\alpha,\infty,4)=\begin{cases}\frac{230\alpha^3-805\alpha^4+736\alpha^5+8}{63} & \text{if } 0\leq\alpha\leq\frac{1}{2}\\ \frac{-1150\alpha+3220\alpha^2-4370\alpha^3+2875\alpha^4-736\alpha^5+169}{63} & \text{if } \frac{1}{2}\leq\alpha<1\end{cases}$$

$$H(\alpha,\infty,5)=\begin{cases}\frac{470\alpha^3-1645\alpha^4+1504\alpha^5+15}{126} & \text{if } 0\leq\alpha\leq\frac{1}{2}\\ \frac{-2350\alpha+6580\alpha^2-8930\alpha^3+5875\alpha^4-1504\alpha^5+344}{126} & \text{if } \frac{1}{2}\leq\alpha<1\end{cases}$$

$$H(\alpha,\infty,6)=\begin{cases}\frac{1690\alpha^3-5915\alpha^4+5408\alpha^5+50}{462} & \text{if } 0\leq\alpha\leq\frac{1}{2}\\ \frac{\left(\substack{-8450\alpha+23\,660\alpha^2-32\,110\alpha^3+\\ 21\,125\alpha^4-5408\alpha^5+1233}\right)}{462} & \text{if } \frac{1}{2}\leq\alpha<1\end{cases}$$

The expressions above can be plotted as follows in Figure 2:

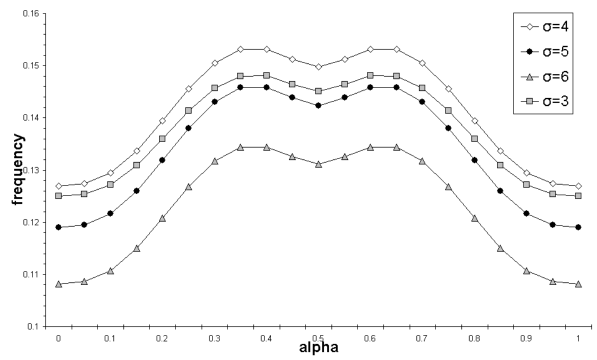

Fig.2 Strategic sponsoring under amendment for a large electorate

The four curves reveal a strict symmetry with respect to line $\alpha=\frac{1}{2}$, a local minimum. Moreover, the four frequency curves have the same two local maxima, at $\alpha=\frac{3}{8}=0.375$ and at $\alpha=\frac{5}{8}=0.625$. In other words, this means that strategic sponsoring is more likely to occur when the quota is 37.5% or 62.5%. We can also note that the frequencies rise between 3 and 4 sponsors and then decrease for 5 and 6 sponsors. At $\alpha=\frac{3}{8}$ or $\frac{5}{8}$,

σ (sponsors)	3	4	5	6
$H(\alpha,\infty,\sigma) =$	0.148	0.153	0.146	0.135

This means that strategic sponsoring is most likely to occur when the number σ of sponsors is fixed to four.

Figure 3 illustrates the following facts: (*i*) frequencies decrease as the number of sponsors rises; (*ii*) there is a strict symmetry with respect to line $\alpha = \frac{1}{2}$, for any number of sponsors; (*iii*) furthermore, the value α at the local maxima remain constant whatever the number of voters and sponsors; this can easily be checked by the reader, by taking the derivative with respect to α and setting it equal to 0.

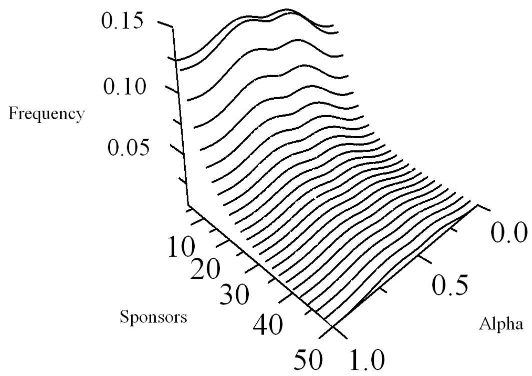

Fig.3 Strategic sponsoring under amendment for a large electorate

To summarize, we have shown that for amendment procedures, the frequency of occurrence is maximal for 4 sponsors and for α equal to $\frac{5}{8}$ or $\frac{3}{8}$. In the context of simple majority contests the frequencies of strategic sponsoring get smaller and smaller as the number of sponsors grows.

4.2 Successive elimination rules with maximin voters

Let $K(\alpha, n, \sigma)$ be the likelihood that under successive elimination rules with α-majority contests and with maximin voters, a strategic sponsoring situation exists with n voters and σ sponsors. As above, we first consider the case $\alpha = \frac{1}{2}$.

Proposition 7 *Let $SEm_{\frac{1}{2}}$ be the voting rule and σ a fixed proportion k of n. Then for $n \geq 2$, $K(\frac{1}{2}, n, \frac{n}{k}) =$*

$$\begin{cases} \dfrac{5kn\left(\substack{-96k^4n^2-804k^4n-1368k^4-462k^3n^3-3878k^3n^2-7276k^3n-1920k^3+187k^2n^4+\\ 1628k^2n^3+3516k^2n^2+1440k^2n+78kn^5+662kn^4+1324kn^3+480kn^2+5n^6+40n^5+60n^4}\right)}{8(2k+n)(3k+n)(4k+n)(5k+n)(n+1)(n+3)(n+5)(k+n)} & \textit{if } n \equiv 0 \mod 2 \\ \dfrac{5k\left(\substack{-96k^4n^2-456k^4n-408k^4-462k^3n^3-2212k^3n^2-2526k^3n+187k^2n^4+\\ 1012k^2n^3+1161k^2n^2+78kn^5+388kn^4+414kn^3+5n^6+20n^5+15n^4}\right)}{8(2k+n)(3k+n)(4k+n)(5k+n)(n+2)(n+4)(k+n)} & \textit{if } n \equiv 1 \mod 2 \end{cases}$$

For some special values of k, we have:

Corollary 3 *Suppose $SEm_{\frac{1}{2}}$ is the voting rule. Then for $n \geq 2$*

$$K(\tfrac{1}{2}, n, \tfrac{n}{5}) = \begin{cases} \dfrac{125n\left(\substack{n^6+86n^5+1609n^4-2086n^3-\\ 90\,890n^2-275\,200n-219\,000}\right)}{8(n+1)(n+3)(n+5)^2(n+10)(n+15)(n+20)(n+25)} & \textit{if } n \equiv 0 \mod 2 \\ \dfrac{125\left(\substack{n^6+82n^5+1326n^4-6076n^3-\\ 61\,495n^2-120\,150n-51\,000}\right)}{8(n+2)(n+4)(n+5)(n+10)(n+15)(n+20)(n+25)} & \textit{if } n \equiv 1 \mod 2 \end{cases}$$

$$K(\tfrac{1}{2}, n, \tfrac{n}{4}) = \begin{cases} \dfrac{5n\left(\substack{5n^6+352n^5+5700n^4+1776n^3-\\ 214\,592n^2-648\,448n-473\,088}\right)}{2(n+1)(n+3)(n+4)(n+5)(n+8)(n+12)(n+16)(n+20)} & \textit{if } n \equiv 0 \mod 2 \\ \dfrac{5\left(\substack{5n^6+332n^5+4559n^4-11\,720n^3-\\ 147\,568n^2-278\,400n-104\,448}\right)}{2(n+2)(n+4)^2(n+8)(n+12)(n+16)(n+20)} & \textit{if } n \equiv 1 \mod 2 \end{cases}$$

$$K(\tfrac{1}{2}, n, \tfrac{n}{3}) = \begin{cases} \dfrac{15n\left(\substack{5n^6+274n^5+3729n^4+6150n^3-\\ 79\,398n^2-248\,616n-162\,648}\right)}{8(n+1)(n+3)^2(n+5)(n+6)(n+9)(n+12)(n+15)} & \textit{if } n \equiv 0 \mod 2 \\ \dfrac{15\left(5n^5+239n^4+2145n^3-8559n^2-31\,374n-11\,016\right)}{8(n+2)(n+4)(n+6)(n+9)(n+12)(n+15)} & \textit{if } n \equiv 1 \mod 2 \end{cases}$$

$$K(\tfrac{1}{2}, n, \tfrac{n}{2}) = \begin{cases} \dfrac{5n\left(\substack{5n^6+196n^5+2132n^4+5464n^3-\\ 17\,536n^2-65\,312n-37\,248}\right)}{4(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)(n+8)(n+10)} & \textit{if } n \equiv 0 \mod 2 \\ \dfrac{5\left(5n^5+166n^4+1207n^3-1234n^2-12\,120n-3264\right)}{(4n+8)(n+4)^2(n+6)(n+8)(n+10)} & \textit{if } n \equiv 1 \mod 2 \end{cases}$$

$$K(\tfrac{1}{2}, n, n) = \begin{cases} \dfrac{5n\left(5n^6+118n^5+909n^4+2490n^3+22n^2-6640n-3288\right)}{8(n+1)^2(n+2)(n+3)^2(n+4)(n+5)^2} & \textit{if } n \equiv 0 \mod 2 \\ \dfrac{5\left(5n^6+98n^5+590n^4+964n^3-1147n^2-2982n-408\right)}{8(n+1)(n+2)^2(n+3)(n+4)^2(n+5)} & \textit{if } n \equiv 1 \mod 2 \end{cases}$$

Graphically we have in Figure 4:

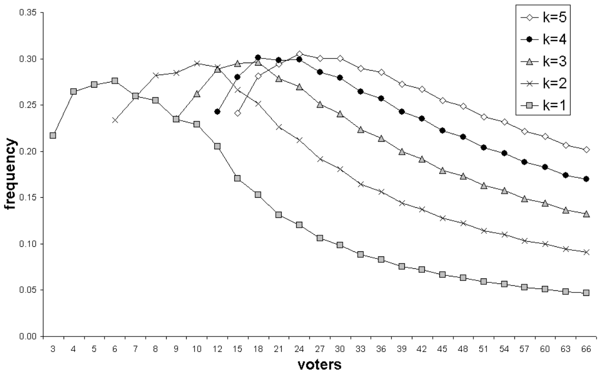

Fig. 4 Strategic sponsoring under maximin with $\dfrac{n}{k}$ sponsors

As for the amendment rules, Figure 4 shows that strategic sponsoring is generally less likely to occur when the number of sponsors is weak relatively to the number of voters. For the french parliament ($n = 577$), the frequencies are 0.016 for $k = 3$ and 0.027 for $k = 5$ (see the appendix for a table of values).

We next come to the general formulae, with $\alpha = \frac{1}{2}$.

Proposition 8 *Let $SEm_{\frac{1}{2}}$ be the voting rule. Then for $n \geq 2$, $K(\frac{1}{2}, n, \sigma) =$*

$$\begin{cases} \dfrac{5\begin{pmatrix} 5n^3\sigma^4+78n^3\sigma^3+187n^3\sigma^2-462n^3\sigma-96n^3+40n^2\sigma^4+662n^2\sigma^3+1628n^2\sigma^2-3878n^2\sigma- \\ 804n^2+60n\sigma^4+1324n\sigma^3+3516n\sigma^2-7276n\sigma-1368n+480\sigma^3+1440\sigma^2-1920\sigma \end{pmatrix}}{8(\sigma+1)(\sigma+2)(\sigma+3)(\sigma+4)(\sigma+5)(n+1)(n+3)(n+5)(\sigma+4)(\sigma+5)(n+1)(n+3)(n+5)} & \text{if } n \equiv 0 \mod 2 \\ \dfrac{5\begin{pmatrix} 5n^2\sigma^4+78n^2\sigma^3+187n^2\sigma^2-462n^2\sigma-96n^2+20n\sigma^4+388n\sigma^3+ \\ 1012n\sigma^2-2212n\sigma-456n+15\sigma^4+414\sigma^3+1161\sigma^2-2526\sigma-408 \end{pmatrix}}{8(\sigma+1)(\sigma+2)(\sigma+3)(\sigma+4)(\sigma+5)(n+2)(n+4)} & \text{if } n \equiv 1 \mod 2 \end{cases}$$

Corollary 4 *In a large society,*

$$K(\tfrac{1}{2}, \infty, \sigma) = \frac{5\left(5\sigma^4+78\sigma^3+187\sigma^2-462\sigma-96\right)}{8(\sigma+1)(\sigma+2)(\sigma+3)(\sigma+4)(\sigma+5)}$$

For some special values of σ, we have:

σ (sponsors)	3	4	5	6
$K(\frac{1}{2}, \infty, \sigma)$	0.252	0.303	0.313	0.307

Again, as for the amendment rule, $K(\frac{1}{2}, n, \sigma)$ tends to 0 when σ tends to infinity. However, it is worth noting that for maximin the maximal occurrence when n tends to infinity is for five sponsors (this number was four for the amendment rules).

As for the amendment rules, we give formulae when n tends to infinity with some special values of σ and present the associated plots (Figure 5 and Figure 6). Formulae for the general case can be found in the associated working paper.

Proposition 9 *Suppose that SEm_α is the voting rule. Then*

$$K(\alpha,\infty,3)=\begin{cases}\frac{180\alpha^2-420\alpha^3+180\alpha^4+108\alpha^5+7}{56} & \text{if } 0\le\alpha\le\frac{1}{2}\\ \frac{-360\alpha+1260\alpha^2-2100\alpha^3+1620\alpha^4-468\alpha^5+55}{56} & \text{if } \frac{1}{2}\le\alpha<1\end{cases}$$

$$K(\alpha,\infty,4)=\begin{cases}\frac{590\alpha^2-1520\alpha^3+1065\alpha^4-62\alpha^5+16}{126} & \text{if } 0\le\alpha\le\frac{1}{2}\\ \frac{-570\alpha+2390\alpha^2-4480\alpha^3+3705\alpha^4-1118\alpha^5+89}{126} & \text{if } \frac{1}{2}\le\alpha<1\end{cases}$$

$$K(\alpha,\infty,5)=\begin{cases}\frac{670\alpha^2-1790\alpha^3+1410\alpha^4-238\alpha^5+15}{126} & \text{if } 0\le\alpha\le\frac{1}{2}\\ \frac{-420\alpha+2050\alpha^2-4150\alpha^3+3570\alpha^4-1102\alpha^5+67}{126} & \text{if } \frac{1}{2}\le\alpha<1\end{cases}$$

$$K(\alpha,\infty,6)=\begin{cases}\frac{1280\alpha^2-3490\alpha^3+2905\alpha^4-624\alpha^5+25}{231} & \text{if } 0\le\alpha\le\frac{1}{2}\\ \frac{-590\alpha+3280\alpha^2-7010\alpha^3+6185\alpha^4-1936\alpha^5+96}{231} & \text{if } \frac{1}{2}\le\alpha<1\end{cases}$$

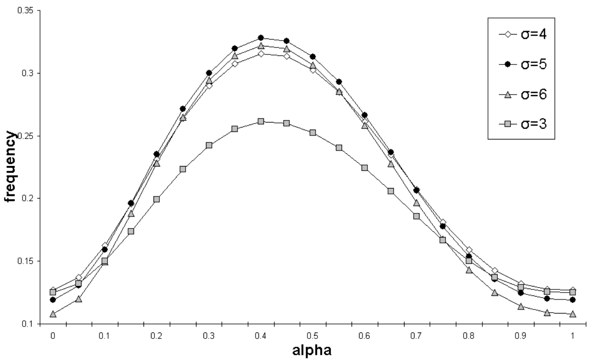

Fig. 5 Strategic sponsoring under maximin for a large electorate

Figure 5 shows that all the curves have their local maxima around $\alpha=\sqrt{2}-1$. However when the number of sponsors rises, the maxima move towards the ordinates axis.

For example for $\sigma=4$, the maximum is obtain at $\alpha=0.414$, and at $\alpha=0.412$ for $\sigma=6$. Further, frequencies are highest for 5 sponsors, and the maximum (32.8 15 %) is reached at $\alpha=0.413$. We can also analyze the evolution of the frequencies when the number of sponsors rises, by studying the three-dimensional plot.

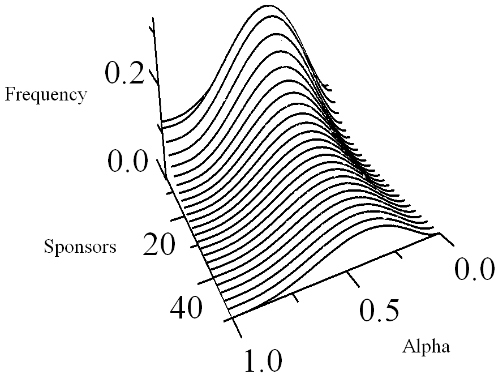

Fig. 6 Strategic sponsoring under maximin for a large electorate

Again, it appears that, as the number of sponsors rises, (i) the trend of frequencies is decreasing and (ii) the local maxima correspond to values of α closer to 0.

Then under successive elimination rules with maximin behavior, strategic sponsoring occurrence is maximal for $\sigma = 5$ at α around $\sqrt{2} - 1$ and decreases with simple majority contests as σ gets larger.

4.3 Successive elimination rules with maximax voters

Let $J(\alpha, n, \sigma)$ be the likelihood of strategic sponsoring situations under successive elimination rules with α-majority contests and with maximax voters a exists. Again, we first consider the $\alpha = \frac{1}{2}$ case.

Proposition 10 *Let $SEM_{\frac{1}{2}}$ be the voting rule. Then for $n \geq 2$, $J(\frac{1}{2}, n, k) =$*

$$\begin{cases} \dfrac{5kn\left(\substack{-72k^4n^2-288k^4n-96k^4-454k^3n^3-2856k^3n^2-5352k^3n-3840k^3+261k^2n^4+\\ 2004k^2n^3+4508k^2n^2+2880k^2n+94kn^5+696kn^4+1512kn^3+960kn^2+3n^6+12n^5+4n^4}\right)}{16(2k+n)(3k+n)(4k+n)(5k+n)(n+1)(n+3)(n+5)(k+n)} & \\ \hfill \textit{if } n \equiv 0 \mod 2 & \\ \dfrac{5k\left(\substack{-72k^4n^2-192k^4n+264k^4-454k^3n^3-1904k^3n^2-1482k^3n+261k^2n^4+\\ 1336k^2n^3+1283k^2n^2+94kn^5+464kn^4+402kn^3+3n^6+8n^5-11n^4}\right)}{16(2k+n)(3k+n)(4k+n)(5k+n)(n+2)(n+4)(k+n)} & \\ \hfill \textit{if } n \equiv 1 \mod 2 & \end{cases}$$

For some special values of k, we have:

Corollary 5 *Suppose that $SEM_{\frac{1}{2}}$. Then for $n \geq 2$*

$$J(\tfrac{1}{2}, n, \tfrac{n}{5}) = \begin{cases} \dfrac{25n\left(3n^6+482n^5+10\,009n^4+910n^3-284\,500n^2-777\,000n-540\,000\right)}{16(n+1)(n+3)(n+5)^2(n+10)(n+15)(n+20)(n+25)} & \text{if } n \equiv 0 \mod 2 \\[2ex] \dfrac{25\left(3n^6+478n^5+8834n^4-21\,340n^3-250\,925n^2-305\,250n+165\,000\right)}{16(n+2)(n+4)(n+5)(n+10)(n+15)(n+20)(n+25)} & \text{if } n \equiv 1 \mod 2 \end{cases}$$

$$J(\tfrac{1}{2}, n, \tfrac{n}{4}) = \begin{cases} \dfrac{5n\left(3n^6+388n^5+6964n^4+9056n^3-125\,248n^2-370\,176n-270\,336\right)}{4(n+1)(n+3)(n+4)(n+5)(n+8)(n+12)(n+16)(n+20)} & \text{if } n \equiv 0 \mod 2 \\[2ex] \dfrac{15\left(n^6+128n^5+2007n^4-2024n^3-39\,920n^2-48\,000n+22\,528\right)}{4(n+2)(n+4)^2(n+8)(n+12)(n+16)(n+20)} & \text{if } n \equiv 1 \mod 2 \end{cases}$$

$$J(\tfrac{1}{2}, n, \tfrac{n}{3}) = \begin{cases} \dfrac{15n\left(3n^6+294n^5+4441n^4+10\,314n^3-39\,492n^2-141\,912n-111\,456\right)}{16(n+1)(n+3)^2(n+5)(n+6)(n+9)(n+12)(n+15)} & \text{if } n \equiv 0 \mod 2 \\[2ex] \dfrac{15\left(3n^6+290n^5+3730n^4+972n^3-45\,693n^2-55\,566n+21\,384\right)}{16(n+2)(n+3)(n+4)(n+6)(n+9)(n+12)(n+15)} & \text{if } n \equiv 1 \mod 2 \end{cases}$$

$$J(\tfrac{1}{2}, n, \tfrac{n}{2}) = \begin{cases} \dfrac{5n\left(3n^6+200n^5+2440n^4+7408n^3-4048n^2-35\,904n-32\,256\right)}{8(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)(n+8)(n+10)} & \text{if } n \equiv 0 \mod 2 \\[2ex] \dfrac{5\left(3n^6+196n^5+1961n^4+2516n^3-11\,252n^2-14\,928n+4224\right)}{8(n+2)^2(n+4)^2(n+6)(n+8)(n+10)} & \text{if } n \equiv 1 \mod 2 \end{cases}$$

$$J(\tfrac{1}{2}, n, n) = \begin{cases} \dfrac{5n\left(3n^6+106n^5+961n^4+3062n^3+2540n^2-2760n-3936\right)}{16(n+1)^2(n+2)(n+3)^2(n+4)(n+5)^2} & \text{if } n \equiv 0 \mod 2 \\[2ex] \dfrac{15(n-1)\left(n^5+35n^4+273n^3+701n^2+470n-88\right)}{16(n+1)(n+2)^2(n+3)(n+4)^2(n+5)} & \text{if } n \equiv 1 \mod 2 \end{cases}$$

We then plot the above frequencies in Figure 7:

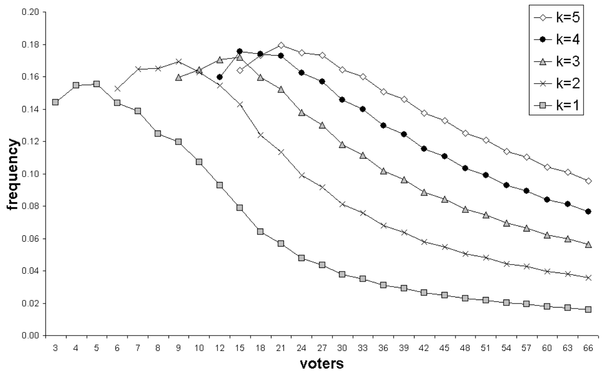

Fig. 7 Strategic sponsoring under maximax with $\dfrac{n}{k}$ sponsors

And as for the two previous procedures, the frequency of coalitional strategic sponsoring, for a given value of n, is generally greater when the number of sponsors is small as compared with the number of voters. For the french parliament($n = 577$), the frequencies are 0.009 for $k = 5$ and 0.002 for $k = 1$ (see the appendix for a table of values).

We now examine the general case for simple majority.

Proposition 11 *Suppose that $SEM_{\frac{1}{2}}$ is the voting rule. Then for $n \geq 2$ and $\sigma \geq 3$, $J(\frac{1}{2}, n, \sigma) =$*

$$\begin{cases} \dfrac{5\left(\substack{3n^3\sigma^4+94n^3\sigma^3+261n^3\sigma^2-454n^3\sigma-72n^3+12n^2\sigma^4+696n^2\sigma^3+2004n^2\sigma^2-\\ 2856n^2\sigma-288n^2+4n\sigma^4+1512n\sigma^3+4508n\sigma^2-5352n\sigma-96n+960\sigma^3+2880\sigma^2-3840\sigma}\right)}{16(\sigma+1)(\sigma+2)(\sigma+3)(\sigma+4)(\sigma+5)(n+1)(n+3)(n+5)} & \text{if } n \equiv 0 \mod 2 \\ \dfrac{5\left(\substack{3n^2\sigma^4+94n^2\sigma^3+261n^2\sigma^2-454n^2\sigma-72n^2+8n\sigma^4+464n\sigma^3+\\ 1336n\sigma^2-1904n\sigma-192n-11\sigma^4+402\sigma^3+1283\sigma^2-1482\sigma+264}\right)}{16(\sigma+1)(\sigma+2)(\sigma+3)(\sigma+4)(\sigma+5)(n+2)(n+4)} & \text{if } n \equiv 1 \mod 2 \end{cases}$$

Corollary 6 *In a large society,*

$$J(\tfrac{1}{2}, \infty, \sigma) = \frac{5\left(3\sigma^4+94\sigma^3+261\sigma^2-454\sigma-72\right)}{16(\sigma+1)(\sigma+2)(\sigma+3)(\sigma+4)(\sigma+5)}$$

For special values of σ, we have:

σ (sponsors)	3	4	5	6
$J(\frac{1}{2}, \infty, \sigma)$	0.172	0.188	0.184	0.174

Once again, the likelihood of strategic sponsoring is maximal for 4 sponsors. Note that $J(\frac{1}{2}, n, \sigma)$ also tends to 0 as σ tends to infinity.

As for the two previous procedures general formulae are given in the associated working paper.

And for some arbitrary values of σ when n tends to infinity, we obtain:

Proposition 12 *Suppose that SEM_α is the voting rule. Then*

$$J(\alpha,\infty,3)=\begin{cases}\frac{180\alpha^3-570\alpha^4+504\alpha^5+7}{56} & \text{if } 0\le\alpha\le\frac{1}{2}\\ \frac{-780\alpha+2040\alpha^2-2460\alpha^3+1350\alpha^4-264\alpha^5+121}{56} & \text{if } \frac{1}{2}\le\alpha<1\end{cases}$$

$$J(\alpha,\infty,4)=\begin{cases}\frac{2\left(115\alpha^3-355\alpha^4+311\alpha^5+4\right)}{63} & \text{if } 0\le\alpha\le\frac{1}{2}\\ \frac{2\left(-480\alpha+1230\alpha^2-1425\alpha^3+725\alpha^4-121\alpha^5+75\right)}{63} & \text{if } \frac{1}{2}\le\alpha<1\end{cases}$$

$$J(\alpha,\infty,5)=\begin{cases}\frac{470\alpha^3-1435\alpha^4+1252\alpha^5+15}{126} & \text{if } 0\le\alpha\le\frac{1}{2}\\ \frac{-1930\alpha+4900\alpha^2-5570\alpha^3+2725\alpha^4-412\alpha^5+302}{126} & \text{if } \frac{1}{2}\le\alpha<1\end{cases}$$

$$J(\alpha,\infty,6)=\begin{cases}\frac{845\alpha^3-2565\alpha^4+2233\alpha^5+25}{231} & \text{if } 0\le\alpha\le\frac{1}{2}\\ \frac{-3440\alpha+8690\alpha^2-9775\alpha^3+4675\alpha^4-663\alpha^5+538}{231} & \text{if } \frac{1}{2}\le\alpha<1\end{cases}$$

And graphically in Figure 8 and Figure 9:

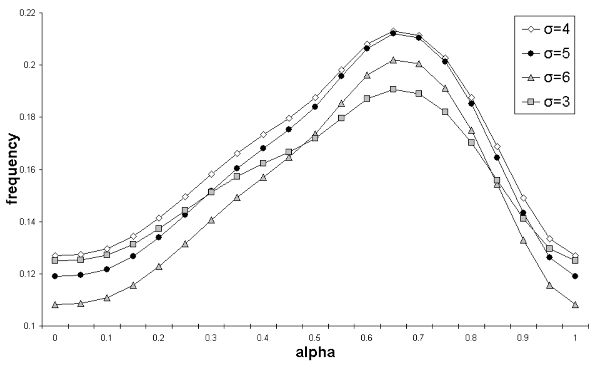

Fig. 8 Strategic sponsoring under maximax for a large electorate

The four frequency curves reveal a local maximum at around $\alpha = 0.66$. But contrary to the maximin case, the maxima move at the further from the ordinates axis as the number of sponsors rises. For example for $\sigma = 3$ and $\sigma = 5$, the maximum is obtain at $\alpha = 0.661$ and at $\alpha = 0.665$, respectively. Moreover the frequency is highest for $\sigma = 4$, and for $\alpha = 0.664$, strategic sponsoring frequencies grow up to 21.335 %.

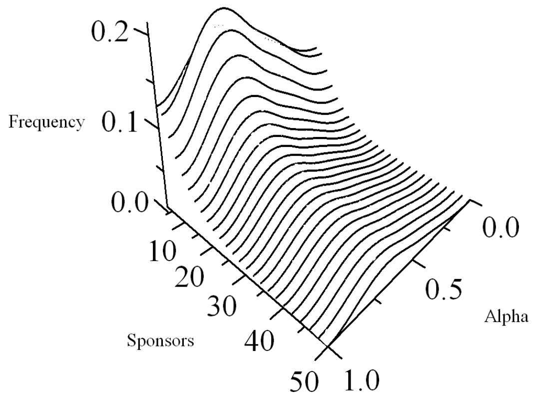

Fig. 9 Strategic sponsoring under maximax for a large electorate

A close look at Figure 9 confirms that for small values of σ, there is a local maximum around $\alpha = 0.66$. But as the number of sponsors gets larger, there is a shift of this maximum towards values of α that are closer to 1.

Then under successive elimination rules with maximax, strategic sponsoring occurrence is maximal when $\sigma = 4$, at α around 0.66 and with $\alpha = \frac{1}{2}$, the smaller the number of sponsors as compared with the number of voters, the higher the frequencies.

5 Concluding discussion

The goal of this paper was to compare three families of parliamentary rules through the study of their vulnerability to strategic sponsoring. In order to do that, we were concerned with computing the frequency of voting situations at which parliamentary rules with $\alpha-$majority contests are vulnerable to coalitional strategic sponsoring of alternatives. Our results show how these frequencies change according to changes in the values of the number of voters, the number of sponsors and the qualified majority. Further, it appears that for all three families of rules: (i) frequencies tend to 0 as the number of sponsors rises, *ceteris paribus*, that is with a fixed number of voters and given any value of α; (ii) however, notice that in actual situations - as emphasized by Dutta and Pattanaik (1978, p. 169) - the number of sponsors is generally very small as compared with the number of voters, and then frequencies are higher.

It also appears that for every possible qualified majority α, and given the number of sponsors and the number of voters, frequencies are always higher with the two versions of successive elimination than with amendment. Now, comparing successive elimination with either maximin or maximax does not lead to a clear conclusion of this kind. However, we observe that for small values of α, frequencies are higher for maximin than for maximax, as illustrated with four sponsors in Figure 10 (see the appendix).

Besides, note that some other results can be derived from the characterizations. We observe that for none of the voting rules under study does the set of strategic sponsoring opportunities coincide with the set of Condorcet cycles. However, for amendment rule, strategic sponsoring Types 3 and 4 requires the presence of a Condorcet cycle, while there is no such relation for the other two rules. In the same way, it can also easily be checked that with single-peaked individual preferences (voters and sponsors), strategic sponsoring occurs under the three procedures.

Finally, it is worth noting that these procedures have been studied in the literature in the context of strategic voting (see Favardin and Lepelley 2006; or Mbih, Moyouwou and Zhao 2006) and it may be interesting to consider some slightly different context by specifically assuming that sponsors participate to the final vote, and examining the possibility of combined strategic sponsoring and strategic voting.

Appendix

In this appendix, we give a figure comparing the three rules for large electorates and we also gives tables summarizing the results given in Section 4.

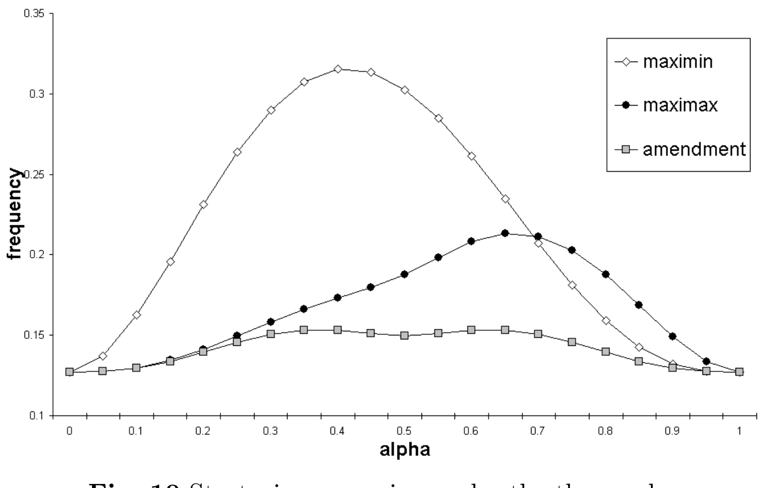

Fig. 10 Strategic sponsoring under the three rules

Table 1 Strategic sponsoring with $\frac{n}{k}$ sponsors with alpha $= 1/2$

Table 1 Strategic sponsoring with $\frac{n}{k}$ sponsors with alpha = 1/2

	Amendment					Successive with maximin					Successive with maximax				
n \ σ	(n/5)	(n/4)	(n/3)	(n/2)	n	(n/5)	(n/4)	(n/3)	(n/2)	n	(n/5)	(n/4)	(n/3)	(n/2)	n
3					0.13648					0.21684					0.14413
4					0.14727					0.26449					0.15445
5					0.13681					0.27217					0.15533
6				0.14378	0.12960				0.23353	0.27568				0.15260	0.14358
7				0.14653	0.11590				0.25901	0.25940				0.16480	0.13872
8				0.14883	0.10715				0.28234	0.25459				0.16500	0.12452
9			0.14298	0.14447	0.09583			0.23514	0.28478	0.23464			0.15947	0.16923	0.11977
10			0.14827	0.14165	0.08838			0.26241	0.29472	0.22893			0.16417	0.16263	0.10711
12		0.14463	0.14929	0.13057	0.07371		0.24281	0.28907	0.29078	0.20525		0.15966	0.17044	0.15440	0.09275
15	0.14416	0.14924	0.14128	0.11253	0.05715	0.24093	0.27982	0.29474	0.26671	0.17077	0.16389	0.17565	0.17198	0.1429	0.07901
18	0.14989	0.14664	0.13083	0.09770	0.04636	0.28121	0.30129	0.29613	0.25169	0.15267	0.17317	0.17379	0.15949	0.12395	0.06404
21	0.14830	0.13911	0.11866	0.08439	0.03804	0.29425	0.29851	0.27896	0.22598	0.13112	0.17941	0.17278	0.15212	0.11348	0.05675
24	0.14416	0.13093	0.10791	0.07398	0.03219	0.30540	0.29881	0.26999	0.21202	0.11996	0.17472	0.16244	0.13795	0.09915	0.04785
27	0.13767	0.12180	0.09759	0.06494	0.02747	0.30046	0.28544	0.25067	0.19153	0.10572	0.17311	0.15708	0.13022	0.09152	0.04359
30	0.13098	0.11343	0.08886	0.05777	0.02397	0.30041	0.27945	0.24059	0.18057	0.09834	0.16437	0.14582	0.11811	0.08109	0.03782
33	0.12368	0.10517	0.08083	0.05153	0.02102	0.28944	0.26424	0.22337	0.16473	0.08834	0.16018	0.13987	0.11157	0.07560	0.03510
36	0.11686	0.09786	0.07404	0.04647	0.01874	0.28514	0.25676	0.21428	0.15626	0.08315	0.15085	0.12966	0.10177	0.06787	0.03108
39	0.11003	0.09093	0.06785	0.04200	0.01677	0.27269	0.24236	0.19971	0.14389	0.07577	0.14608	0.12419	0.09649	0.06385	0.02922
42	0.10382	0.08483	0.06256	0.03831	0.01519	0.26695	0.23501	0.19195	0.13727	0.07194	0.13734	0.11540	0.08861	0.05798	0.02630
45	0.09780	0.07911	0.05775	0.03502	0.01381	0.25475	0.22212	0.17974	0.12743	0.06629	0.13273	0.11065	0.08438	0.05495	0.02496
48	0.09237	0.07406	0.05359	0.03225	0.01267	0.24876	0.21540	0.17321	0.12216	0.06335	0.12490	0.10320	0.07799	0.05037	0.02274
51	0.08719	0.06935	0.04979	0.02975	0.01165	0.23743	0.20409	0.16296	0.11419	0.05889	0.12071	0.09916	0.07459	0.04804	0.02175
54	0.08252	0.06518	0.04649	0.02762	0.01079	0.23167	0.19810	0.15745	0.10991	0.05657	0.11383	0.09285	0.06936	0.04439	0.02001
57	0.07808	0.06129	0.04345	0.02568	0.01002	0.22139	0.18823	0.14878	0.10334	0.05297	0.11011	0.08943	0.06658	0.04255	0.01924
60	0.07407	0.05783	0.04077	0.02400	0.00936	0.21606	0.18295	0.14410	0.09982	0.05109	0.10410	0.08406	0.06224	0.03958	0.01784
63	0.07028	0.05458	0.03830	0.02245	0.00875	0.20682	0.17432	0.13670	0.09432	0.04812	0.10083	0.08116	0.05995	0.03811	0.01723
66	0.06684	0.05168	0.03611	0.02111	0.00823	0.20195	0.16968	0.13270	0.09137	0.04657	0.09556	0.07657	0.05631	0.03565	0.01608
577	0.00395	0.00298	0.00209	0.00129	0.00059	0.02707	0.02163	0.01622	0.01081	0.00540	0.00909	0.00713	0.00523	0.00341	0.00166

References

Banks, J.S. (1985). Sophisticated Voting Outcomes and Agenda Control. Social Choice and Welfare, 1(12), 295-306.
Barbera, S., Coelho, D.(2007): On the rule of k names. Working Paper 264 CREA-Barcelona Economics.
Besley T., Coate S. (1997). An economic model of representative democracy. Quarterly Journal of Economics, 112(2), 85-114.
Courtin, S., Mbih, B., Moyouwou, I.(2008): Sponsoring under parliamentary voting with anonymous voters. Working Paper CREM-Université de Caen.
Dutta B., Pattanaik P. K. (1978). On strategic manipulation of issues in group decision. In North Holland (Ed.), Strategy and group choice. Amsterdam: Economica.
Dutta B., Jackson O., Le Breton M. (2000). Voting by Successive Elimination and Strategic Candidacy. Journal of Economic Theory, 103(3), 190-218.
Dutta B., Jackson O., Le Breton M. (2001). Strategic candidacy and voting procedures. Econometrica, 69(7), 1013-1037.
Favardin, P., Lepelley, D. (2006). Some further results on the manipulability of social choice rules. Social Choice and Welfare, 26(6), 485–509.
Gehrlein W. V., Fishburn P.C. (1976). The probability of the paradox of voting: a computable solution. Journal of Economic Theory, 13(8), 14-25.
Gibbard, A. F. (1973). Manipulation of voting schemes: a general result. Econometrica, 41(7), 587-601.
Kuga K., Nagatani H. (1974). Voter antagonism and the paradox of voting. Econometrica, 42(11), 1045-67.
Majumdar, P. (1956). Choice and revealed preference. Econometrica, 24(1), 71-73.
Mbih, B., Moyouwou, I., Picot, J. (2008). Pareto violations of parliamentary voting systems. Economic theory, 34(2), 331-358.
Mbih, B., Moyouwou, I., Zhao, X. (2006): On the responsiveness of parliamentary social choice functions. Working Paper CREM-Université de Caen.
Miller, N. R. (1995). Committees, Agendas, and Voting. Reading: Harwood Academic.
Osborne M. J., Slivinski A. (1996). A model of political competition with citizen-candidates. The Quarterly Journal of Economics, 111(2), 65-96.
Pattanaik, P. K.(1978). Strategy and group choice. Amsterdam: Economica.
Rasch, B., E. (1995). Parliamentary Voting Rules. In Herbert Döring (Ed.), Parliaments and Majority Rule in Western Europe. Frankfurt: Campus Verlag.
Satterthwaite, M. (1975). Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions. Journal of Economic Theory, 10(4), 187-217.
Slutsky, S.(1979). Equilibrium under $\alpha-$majority voting, Econometrica, 47(10), 1113–1125.